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A morphism of Green functors

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1 Introduction

This article is a survey of [Od07]. Bouc introduced the Dress construction for a *Green functor* ([Bo03a] Theorem 5.1): If A is a Green functor for G over a commutative ring \mathcal{O} , and Γ is a crossed G -monoid, then the Mackey functor A_Γ obtained by the Dress construction has a natural structure of a Green functor, and its evaluation $A_\Gamma(G)$ is an \mathcal{O} -algebra. Bouc's construction involves as special cases the construction of the crossed Burnside ring obtained from the Burnside ring Green functor, the Hochschild cohomology ring of G obtained from the group cohomology Green functor, and the Grothendieck ring of the Drinfel'd double of G obtained from the Grothendieck ring Green functor for a group algebra. We also point out that Bouc's construction is discussed in [Wi04]. In this paper, we obtain an induction theorem for the Drinfel'd double for G by using a formula for the primitive idempotents of the crossed Burnside ring [OY01], Bouc's construction, and some properties of Witherspoon's Green functor $R(D_G(*))$. The theorem implies Artin's induction theorem for a group algebra over \mathbb{C} . This is a new proof of Artin induction theorem.

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2 Crossed G -sets

(2.1) Notation. Let G be a finite group. If H is a subgroup of G , and $g \in G$, the conjugate subgroup gHg^{-1} of G is denoted by gH . The normalizer of H in G is denoted by $N_G(H)$. The centralizer of H (resp. $g \in G$) in G is denoted by $C_G(H)$ (resp. $C_G(g)$). A set of representatives in G of G/H is denoted by $[G/H]$. If X is a G -set, the stabilizer in G of element x of X is denoted by G_x . If X and Y are G -sets, the intersection $G_x \cap G_y$ of stabilizers in G of element (x, y) of $X \times Y$ is denoted by $G_{x,y}$. The set of orbits of H on X is denoted by $H \backslash X$, and $[H \backslash X]$ denotes a set of representatives in X of $H \backslash X$.

(2.2) Crossed Burnside rings. Let G be a finite group. In [Bo03a], Bouc defined a crossed G -monoid as follows. A *crossed G -monoid* (Γ, φ) is a pair consisting of a finite monoid Γ with a left action of G by monoid automorphisms (denoted by $(g, \gamma) \mapsto g\gamma$ or $(g, \gamma) \mapsto {}^g\gamma$, for $g \in G$ and $\gamma \in \Gamma$), and a map of G -monoids φ from Γ to the G -set

G^c with G -action defined by conjugation (i.e. a map φ which is both a map of monoids and a map of G -sets). In this paper, since we use only the trivial crossed G -monoid $(\Gamma, \varphi) = (G^c, id_{G^c})$, we denote by Γ or G^c a crossed G -monoid. A *crossed G -set* (X, α) over a crossed G -monoid Γ , is a pair consisting of a finite G -set X , together with a map α of G -sets from X to Γ . A *morphism* of crossed G -sets from (X, α) to (Y, β) is a G -map f from X to Y such that $\beta \circ f = \alpha$. Crossed G -sets over Γ and crossed G -maps make a category $G\text{-xset}/\Gamma$. The *tensor product* of crossed G -sets (X, α) and (Y, β) is defined by $(X \times Y, \alpha.\beta)$, where $X \times Y$ is the direct product of X and Y , with diagonal G -action, and $\alpha.\beta$ is the map from $X \times Y$ to G^c defined by $\alpha.\beta(x, y) = \alpha(x)\beta(y)$. We denote by $X\Omega(G, \Gamma)$ the Grothendieck ring of the category $G\text{-xset}/\Gamma$ with respect to disjoint union and tensor product. We call it the *crossed Burnside ring*. The crossed Burnside ring $G\text{-xset}/1^c$ over the crossed 1^c -monoid is the ordinary Burnside ring $B(G)$. Since any crossed G -set is a disjoint union of *transitive* crossed G -sets (see 2.12 of [OY01]), $G\text{-xset}/\Gamma$ has the following free \mathbb{Z} -basis as an abelian group:

$$\{(G/D)_s \mid D \in [G \setminus S(G)], s \in [G \setminus C_\Gamma(D)]\}.$$

If Γ is a normal subgroup of G or an abelian group, then a formula for the primitive idempotents of $\mathbb{K}X\Omega(G, \Gamma)$ over a splitting field \mathbb{K} of characteristic 0 has been given by Oda and Yoshida (see Lemma (5.5) of [OY01]).

(2.3) Theorem. [OY01] *Let \mathbb{K} be a field of characteristic 0 which is a splitting field for all subgroups of G .*

(1) *For $H \leq G$ and an irreducible \mathbb{K} -character θ of $C_\Gamma(H)$, we put*

$$e_{H,\theta} = \frac{\theta(1)}{|N_G(H)||C_\Gamma(H)|} \sum_{D \leq H} \sum_{s \in C_\Gamma(H)} |D| \mu(D, H) \tilde{\theta}(s^{-1})(G/D)_s,$$

where $\tilde{\theta}$ is the sum of all distinct $N_G(H)$ -conjugates of θ . Then

$$\{e_{H,\theta} \mid H \in [G \setminus S(G)], \theta \in [N_G(H) \setminus \text{Irr}_{\mathbb{K}}(C_\Gamma(H))]\}$$

is a set of orthogonal idempotents of the crossed Burnside ring $\mathbb{K}X\Omega(G, \Gamma)$ over \mathbb{K} such that

$$(G/G)_{1_G} = 1_{\mathbb{K}X\Omega(G, \Gamma)} = \sum_{H, \theta} e_{H,\theta}.$$

Moreover, the idempotents $e_{H,\theta}$ are all primitive and conversely any primitive idempotent of $\mathbb{K}X\Omega(G, \Gamma)$ has this form.

A formula for the primitive idempotents of $\mathcal{O}X\Omega(G, G^c)$ over a p -local ring \mathcal{O} has been given by Bouc [Bo03b].

3 Bouc's constructions of Green functors

(3.1) Burnside Green functors. We recall the crossed Burnside ring Green functor $X\Omega(*, G^c)$ in terms of subgroups of G (see 4.1 of [OY04]). Let $S(H)$ be the family of all subgroups of $H \leq G$ and $C_G(D)$ the centralizer of $D \leq H$. Then the assignment

$$H(\leq G) \longmapsto X\Omega(H, G^c) = \langle (H/D)_s \mid D \in [H \setminus S(H)] s \in [H \setminus C_G(D)] \rangle_{\mathbb{Z}}$$

gives a Green functor for G over \mathbb{Z} equipped with

$$\begin{aligned} \text{ind}_L^H &: X\Omega(L, G^c) \longrightarrow X\Omega(H, G^c) &: (L/D)_s \longmapsto (H/D)_s, \\ \text{res}_L^H &: X\Omega(H, G^c) \longrightarrow X\Omega(L, G^c) &: (H/D)_s \longmapsto \sum_{g \in [L \setminus H/D]} (L/L \cap {}^g D)_{gs}, \\ \text{con}_{H,g} &: X\Omega(H, G^c) \longrightarrow X\Omega({}^g H, G^c) &: (H/D)_s \longmapsto ({}^g H/{}^g D)_{gs}, \end{aligned}$$

where $D \leq L \leq H \leq G$ and $g \in G$. In order to note the Green functor structure of $X\Omega(*, G^c)$, we shall discuss briefly an equivalence between the category $G\text{-set} \downarrow_{(G/H \times G^c)}$ of finite G -sets over the G -set $G/H \times G^c$ (see 2.4 of [Bo97]) and the category $H\text{-set} \downarrow_{G^c}$ of finite H -sets over the H -set G^c with the H -action defined by conjugation. Let Ω be the Burnside Green functor for G over \mathbb{Z} in terms of G -sets. By Proposition 2.4.2 of [Bo97], $\Omega_{G^c}(G/H) = \Omega((G/H) \times G^c)$ is isomorphic to the Grothendieck group of $G\text{-set} \downarrow_{(G/H \times G^c)}$, with relations given by decomposition into disjoint union. It is easy to see that the G -sets

$$[K, s] : G/K \rightarrow G/H \times G^c : gK \mapsto (gH, {}^g s)$$

over $G/H \times G^c$, for $K \in [H \setminus S(H)]$ and $s \in [H \setminus C_G(K)]$, form a basis of $\Omega(G/H \times G^c)$ over \mathbb{Z} . We denote by $(G/K, [K, s])$ an element of the basis of $\Omega(G/H \times G^c)$. Theorem 5.1 of [Bo03a] shows that Ω_{G^c} is a Green functor. If $(G/K, [K, s])$ and $(G/L, [L, t])$ are elements of the basis of $\Omega(G/H \times G^c)$, then we have the following commutative diagram

$$\begin{array}{ccccc} \bigsqcup_{x \in K \setminus H/L} G/K \cap {}^x L & \xrightarrow{\sqcup_x \pi_K^K \cap {}^x L \times (\pi_K^K \cap {}^x L \cap {}^x L, {}^x)} & G/K \times G/L & \xrightarrow{[K, s] \times [L, t]} & (G/H \times G^c) \times (G/H \times G^c) \\ \downarrow \sqcup_x [K \cap {}^x L, s \cdot {}^x t] & \text{P.B.} & \downarrow & & \downarrow f \\ G/H \times G^c & \xrightarrow{\delta_{G/H} \times \text{id}_{G^c}} & G/H \times G/H \times G^c & \xrightarrow{\text{id}} & G/H \times G/H \times G^c, \end{array}$$

where the map f from $G/H \times G^c \times G/H \times G^c$ to $G/H \times G/H \times G^c$ maps $(xK, \gamma_1, yL, \gamma_2)$ to $(xK, yL, \gamma_1 \gamma_2)$ (see section 5 of [Bo03a]). The left square is a pullback square. Theorem 5.1 of [Bo03a] shows that the product of $(G/K, [K, s])$ and $(G/L, [L, t])$ on $\Omega(G/H \times G^c)$ is given by

$$(G/K, [K, s]) \cdot (G/L, [L, t]) = \sum_{x \in [K \setminus H/L]} (G/K \cap {}^x L, [K \cap {}^x L, s \cdot {}^x t]). \quad (3.1.1)$$

We have a functor F mapping $G\text{-set} \downarrow_{(G/H \times G^c)}$ to $H\text{-set} \downarrow_{G^c}$ defined for a transitive G -set $[K, s] : G/K \rightarrow G/H \times G^c$ over $G/H \times G^c$ by

$$F : (G/K, [K, s]) \mapsto \langle K, s \rangle : [K, s]^{-1}(\{H\} \times G^c) \rightarrow G^c$$

as in Lemma 2.4.1 of [Bo97]. We also denote by $[K, s]$ the H -map $H/K \rightarrow G^c$ defined by $gK \mapsto {}^g s$. It is clear that the H -sets

$$[K, s] : H/K \rightarrow G^c : gK \mapsto {}^g s$$

over the H -set G^c , for $K \in [H \setminus S(H)]$ and $s \in [H \setminus C_G(K)]$, form a basis of $\Omega \downarrow_H^G(G^c)$ over \mathbb{Z} , where $\Omega \downarrow_H^G$ is a Green functor for H given by the restriction to H of G . We denote by $(H/K, [K, s])$ this element of the basis of $\Omega \downarrow_H^G(G^c)$. It is easy to see that F gives an equivalence of categories from $G\text{-set} \downarrow_{(G/H \times G^c)}$ to $H\text{-set} \downarrow_{G^c}$ for any subgroup H of G . The

inverse equivalence is given by the induction functor from $H\text{-set} \downarrow_{G^c}$ to $G\text{-set} \downarrow_{(G/H \times G^c)}$. The images of (3.1.1) under F are

$$(H/K, [K, s]) \cdot (H/L, [L, t]) = \sum_{x \in [K \setminus H/L]} (H/K \cap {}^x L, [K \cap {}^x L, s \cdot {}^x t]).$$

in $H\text{-set} \downarrow_{G^c}$. The Grothendieck group of $H\text{-set} \downarrow_{G^c}$ is isomorphic to $X\Omega(H, G^c)$. We can define a product

$$(H/K)_s \cdot (H/L)_t = \sum_{x \in [K \setminus H/L]} (H/K \cap {}^x L)_{s \cdot {}^x t}$$

for any two elements $(H/K)_s$ and $(H/L)_t$ of the basis of $X\Omega(H, G^c)$. It is clear that the element $(H/H)_{1_G}$ for the identity element 1_G of G is the identity element of $X\Omega(H, G^c)$. This gives a unitary ring structure to $X\Omega(H, G^c)$ for a subgroup H of G .

(3.2) Witherspoon's Green functor. Witherspoon introduced a Green functor $R_{\mathbb{C}}(D_G(*))$ for G over \mathbb{Z} (see [Wi96] Section 5). For each subgroup H of G , there is a subalgebra

$$D_G(H) = \sum_{g \in G, h \in H} \mathbb{C} \phi_g h$$

of the Drinfel'd (quantum) double $D(G)$ of $\mathbb{C}G$ [Dr86], where ϕ_g is an element of the basis $\{\phi_g\}_{g \in G}$ of the dual space $(\mathbb{C}G)^* = \text{Hom}_{\mathbb{C}}(\mathbb{C}G, \mathbb{C})$. Note that $D_G(G) = D(G)$ and $R(D(G))$ is the representation ring of $D(G)$ or equivalently the Grothendieck ring of Hopf bimodules for the Hopf algebra $\mathbb{C}G$ ([Ro95], [Bo03a], [OY04]). Let $R_{\mathbb{C}}(D_G(H))$ be the Grothendieck (representation) ring of $D_G(H)$ for subgroup H of G . Then the assignment

$$H \longmapsto R_{\mathbb{C}}(D_G(H))$$

where $H \leq G$ gives a Green functor for G over \mathbb{Z} with operations given by

$$\begin{aligned} \text{Dres}_L^H &: R_{\mathbb{C}}(D_G(H)) \longrightarrow R_{\mathbb{C}}(D_G(L)) &: U &\longmapsto U \downarrow_{D_G(L)}, \\ \text{Dind}_L^H &: R_{\mathbb{C}}(D_G(L)) \longrightarrow R_{\mathbb{C}}(D_G(H)) &: V &\longmapsto D_G(H) \otimes_{D_G(L)} V, \\ \text{Dconj}_{H,g} &: R_{\mathbb{C}}(D_G(H)) \longrightarrow R_{\mathbb{C}}(D_G({}^g H)) &: U &\longmapsto {}^g U = g D_G(H) \otimes_{D_G(H)} U, \end{aligned}$$

where $U \downarrow_{D_G(L)}$ is a $D_G(L)$ -module by restriction of the action from $D_G(H)$ to $D_G(L)$, $L \leq H \leq G$ and $g \in G$. We use the equivalence of the category of H -vector bundles on G^c with the category of $D_G(H)$ -modules (see [Wi96] Section 2).

(3.3) A morphism of Green functors. Let Ω be the Burnside ring Green functor for G over \mathbb{Z} (see [Bo97] 2.4.2):

- If X is a finite G -set, then $\Omega(X)$ is the Grothendieck ring of the category of finite G -sets over X , where the relations are given by decomposition into disjoint union and product of G -sets.
- If $X \rightarrow X'$ is a G -map, then $\Omega_*(f) : \Omega(X) \rightarrow \Omega(X')$ is defined by $\Omega_*(f)((Y, \phi)) = (Y, f\phi)$ for any G -set $(Y, \phi) = Y \xrightarrow{\phi} X$ over X .

- If $X' \rightarrow X$ is a G -map, then $\Omega^*(f) : \Omega(X) \rightarrow \Omega(X')$ is defined by $\Omega^*(f)((Y, \phi)) = (Y', \phi')$, where (Y', ϕ') is the pull-back of (Y, ϕ) along f , obtained by filling the cartesian square

$$\begin{array}{ccc} Y' & \xrightarrow{a} & Y \\ \phi' \downarrow & & \downarrow \phi \\ X' & \xrightarrow{f} & X. \end{array}$$

Suppose that $R_{\mathbb{C}}$ is the \mathbb{C} -representation (character ring) Green functor for G over \mathbb{Z} , defined on subgroups of G . Then setting $R_{\mathbb{C}}(G/H) = R_{\mathbb{C}}(H)$ leads by linearity to a definition of G -equivariant \mathbb{C} -vector bundles $R_{\mathbb{C}}(X)$ on a G -set X (see [Wi96] Section 2) by using Remark 2.3 of [Bo03a]:

- If X is a finite G -set, then $R_{\mathbb{C}}(X)$ is the Grothendieck ring of the category of G -equivariant \mathbb{C} -vector bundles on the G -set X , for relations given by decomposition into direct sum of vector bundles, the ring structure being induced by the tensor product of vector bundles: one can set

$$R_{\mathbb{C}}(X) = \left(\bigoplus_{x \in X} R_{\mathbb{C}}(G_x) \right)^G$$

where the exponent denotes fixed points under the natural action of G on $\bigoplus_{x \in X} R_{\mathbb{C}}(G_x)$ by permutation of the components, and G_x is the stabilizer of x in G .

- If $f : X \rightarrow X'$ is a G -map, then $R_{\mathbb{C}*}(f) : R_{\mathbb{C}}(X) \rightarrow R_{\mathbb{C}}(X')$ is defined by

$$R_{\mathbb{C}*}(f)(u)_y = \sum_{x \in [G_y \backslash f^{-1}(y)]} t_{G_x}^{G_y}(u_x),$$

where $t_{G_x}^{G_y}$ is the induction map from $R_{\mathbb{C}}(G_x)$ to $R_{\mathbb{C}}(G_y)$, $u \in R_{\mathbb{C}}(X)$, and $y \in X'$.

- If $f : X' \rightarrow X$ is a G -map, then $R_{\mathbb{C}}^*(f) : R_{\mathbb{C}}(X) \rightarrow R_{\mathbb{C}}(X')$ is defined by

$$R_{\mathbb{C}}^*(f)(v)_x = r_{G_{f(x)}}^{G_x}(v_{f(x)}),$$

where $r_{G_{f(x)}}^{G_x}$ is the restriction map from $R_{\mathbb{C}}(G_x)$ to $R_{\mathbb{C}}(G_{f(x)})$, $v \in R_{\mathbb{C}}(X)$, and $x \in X'$.

- The product of the elements $a \in R_{\mathbb{C}}(X)$ and $b \in R_{\mathbb{C}}(Y)$ is defined by

$$(a \times b)_{x,y} = r_{G_{(x,y)}}^{G_x}(a_x) \cdot r_{G_{(x,y)}}^{G_y}(b_y).$$

If X is a finite G -set, denote the natural morphism

$$\theta : \Omega \rightarrow R_{\mathbb{C}}$$

of Green functors defined by the maps $\theta(X) : \Omega(X) \rightarrow R_{\mathbb{C}}(X)$ by

$$(Y, \varphi) = (\varphi : Y \rightarrow X) \longmapsto \{\mathbb{C}[\varphi^{-1}(x)]\}_{x \in X},$$

where $\mathbb{C}[\varphi^{-1}(x)]$ is the permutation module associated to the G_x -set $\varphi^{-1}(x)$.

The following theorem is essential in the proof of Theorem 4.1 of this paper.

(3.4) Theorem (Bouc [Bo03a] 5.1). *Let A be a Green functor for G over a commutative ring \mathcal{O} , Γ a crossed G -monoid, and ε_A an element of $A(\bullet)$ such that for any G -set X and for any $a \in A(X)$*

$$A_*(p_X)(a \times \varepsilon_A) = a = A_*(q_X)(\varepsilon_A \times a)$$

denoting by p_x (resp. q_x) the bijective projection from $X \times \bullet$ (resp. from $\bullet \times X$) to X (see 1.2.1 of [Bo03a]). Then the functor A_Γ is a Green functor for G over \mathcal{O} , with unit ε_{A_Γ} , where ε_{A_Γ} is the element $A_\left(\begin{smallmatrix} \bullet \\ \downarrow \\ 1_G \end{smallmatrix}\right)(\varepsilon_A)$ of $A(\Gamma) = A_\Gamma(\bullet)$. Moreover the correspondence $A \mapsto A_\Gamma$ is an endo-functor of the category of Green functors for G over \mathcal{O} .*

(3.5) Lemma. *Let Ω be the Burnside ring Green functor and G^c the crossed G -monoid. Then there is an isomorphism of Green functors*

$$X\Omega(*, G^c) \cong \Omega_{G^c}.$$

We will denote by $\mathbb{C}[X]$ the \mathbb{C} -permutation module associated to a set X . The endo-functor of the category of Green functors of Theorem (3.4) applied to the morphism θ from Ω to $R_{\mathbb{C}}$ leads to the following lemma.

(3.6) Lemma. *Let $\theta : \Omega \rightarrow R_{\mathbb{C}}$ be the natural morphism from the Burnside Green functor to the Grothendieck ring Green functor. Then the morphism $\theta_{G^c} : \Omega_{G^c} \rightarrow R_{\mathbb{C}G^c}$ given by the Bouc's construction is a morphism of Green functors.*

(3.7) Lemma. *There is a morphism*

$$\theta_{G^c} : X\Omega(*, G^c) \rightarrow R_{\mathbb{C}}(D_G(*))$$

of Green functors.

Let $(H/L)_g$ be an element of the basis of $X\Omega(H, G^c)$. Then the previous lemma shows that $\theta_{G^c}((H/L)_g)$ is an H -vector bundle on G^c . We denote by $[H/L]_g$ this H -vector bundle. Lemma (3.5) shows the following lemma.

(3.8) Lemma. *The H -vector bundle $[H/L]_g$ is the $\mathbb{C}C_H({}^xg)$ -module $\mathbb{C}[({}^xL, {}^xg)^{-1}({}^xg)]$ in the xg -component, for $x \in [H/C_H(g)]$, and 0 in all other components.*

We recall the maps $\text{Incl}_{J,h} : R_{\mathbb{C}}(J) \rightarrow R_{\mathbb{C}}(D_G(J))$, where J is a subgroup of G and $h \in C_G(J)$, introduced in Section 2 of [Wi96]: Given a $\mathbb{C}J$ -module V , $\text{Incl}_{J,h}(V)$ is the $D_G(J)$ -module which is V in the h -component and 0 elsewhere.

(3.9) Lemma. Let θ_{G^c} be the ring homomorphism $\theta_{(G/G) \times G^c}$ from the crossed Burnside ring $X\Omega(G, G^c)$ to the Grothendieck ring $R_{\mathbb{C}}(D_G(G))$ given by the previous lemma. Then the $D(G)$ -module corresponding to the G -vector bundle $\theta_{G^c}((G/L)_g)$ is the induced module

$$D(G) \otimes_{D_G(L)} \text{Incl}_{L,g}(\mathbb{C}[L/L]).$$

(3.10) Sub-Green functors. There is a sub-Green functor $X\Omega(*, G^c)_1$ which assigns to each subgroup H of G the subring $X\Omega(H, G^c)_1$ of $X\Omega(H, G^c)$ generated by the elements $(H/L)_{1_G}$. There is also a sub-Green functor $R_{\mathbb{C}}(D_G(*))_1$ which assigns to each subgroup H of G the subring $R_{\mathbb{C}}(D_G(H))_1$ of $R_{\mathbb{C}}(D_G(H))$ generated by $\text{Incl}_{H,1_G}(V)$'s, where $\text{Incl}_{H,1_G}$ is a functor embedding the category of $\mathbb{C}H$ -modules as a full subcategory of the category of $D_G(H)$ -modules (see, [Wi96] Section 1) and V is a $\mathbb{C}H$ -module. It is easy to see that $X\Omega(H, G^c)_1$ is isomorphic to the Burnside ring $\Omega(H)$ and $R_{\mathbb{C}}(D_G(H))_1$ is isomorphic to the ordinary character ring $R_{\mathbb{C}}(H)$. The homomorphism $\theta_{G^c} \downarrow_{X\Omega(H, G^c)_1}$ is the natural ring homomorphism from $\Omega(H)$ to $R_{\mathbb{C}}(H)$.

(3.11) Characters. Witherspoon pointed out the character of a $\mathbb{C}D(G)$ -module in [Wi96], that appeared in [Lu87]. For $g \in G$ and an irreducible character ρ of $C_G(g)$, a character $\chi_{g,\rho}$ of a $\mathbb{C}D(G)$ -module $U = \{U_h\}_{h \in G^c}$ is given by the formula

$$\chi_{g,\rho}(U) = \frac{1}{\deg \rho} \sum_{h \in C_G(g)} \text{Tr}(g, U_h) \rho(h).$$

The characters of the crossed Burnside ring have been considered by Oda and Yoshida ([OY01], Section 5). For a subgroup H of G and an irreducible character θ of $C_G(H)$, the linear map $\omega_{H,\theta}$ of $X\Omega(G, G^c)$ to \mathbb{C} is the composite of Burnside homomorphism φ_H and a central character $\tilde{\omega}_{H,\theta}$: given a crossed G -set X over G^c , $H \leq G$, and an irreducible character ρ of the group algebra $\mathbb{C}C_G(H)$, $\omega_{H,\rho}(X) = \tilde{\omega}_{H,\rho} \circ \varphi_H(X)$.

For each $h \in G^c$ the h -component of the crossed G -set (X, α) is defined by

$$X[h] = \{x \in X \mid \alpha(x) = h\}.$$

(3.12) Lemma. Let g be an element of G , ρ an irreducible character of $\mathbb{C}C_G(g)$, and θ_{G^c} the homomorphism from $X\Omega(G, G^c)$ to $R_{\mathbb{C}}(D(G))$. Then $\chi_{g,\rho} \theta_{G^c} = \omega_{\langle g \rangle, \rho}$, where $\langle g \rangle$ is the cyclic subgroup generated by g .

4 Induction theorems

The proof of the following theorem is similar to the proof of Theorem 3.5.2 in [Bo00].

(4.1) Theorem. Let G be a finite group. Then

$$\mathbb{C}R_{\mathbb{C}}(D(G)) = \sum_{H \in \mathcal{C}(G)} \text{Dind}_H^G \mathbb{C}R_{\mathbb{C}}(D_G(H)),$$

where $\mathcal{C}(G)$ is the family of cyclic subgroups of G . In other words, any complex character of $D(G)$ is a linear combination with rational coefficients of characters induced from cyclic groups of G .

The previous theorem and (3.10) show the following corollary.

(4.2) **Corollary (Artin).** *Let G be a finite group. Then*

$$\mathbb{Q}R_{\mathbb{C}}(G) = \sum_{H \in \mathcal{C}(G)} \text{Ind}_H^G \mathbb{Q}R_{\mathbb{C}}(H).$$

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